Épreuve de section européenne

The harmonic series

When you add up the first n terms of a sequence you always get a finite number, that's obvious, and this sum is usually called S_n. However, when you sum up an infinite number of terms of a sequence you get a *series* and there are two possible scenarios. The limit, which is the number the sum approaches as more and more terms are added, is either a finite number or not. If the limit is finite, the series is called convergent. If not, it is called divergent.

Let's take a look at the harmonic series: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$

Each term in the series gets smaller and smaller, so you would think that the sum of all the terms would be bounded by a fixed amount. But, strangely enough, the harmonic series is divergent.

Here is Nicolhas Oresme's proof dating back to 1350:
Let's compare the terms of the harmonic series in groups of two, four, eight and so on, starting from the third term.
3rd and 4th terms: $\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
5th to 8th terms: $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4 \times \frac{1}{8} = \frac{1}{2}$.
If we carry on doubling the number of terms, we can see that we will be able to add up these terms to make a value larger than a half.
As a consequence, the harmonic series is infinite! Q.E.D.

Adapted from *Alex's adventures in numberland* by Alex Bellos

Questions

1. What is a series? Give an example.
2. What is a divergent series?
3. What is the expression of the general term a_n which is repeatedly added to give the harmonic series?
4. Using the sequence (a_n), explain why each term in the harmonic series gets smaller and smaller.
5. Now explain the paradox of the harmonic series.
6. (a) The text says that $\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{2}$. In a similar way, show that the sum of the terms of the next group is greater than $\frac{1}{2}$.
   (b) Explain why the harmonic series is divergent.