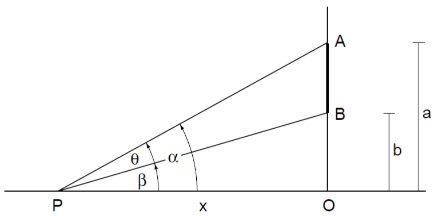
Épreuve de section européenne

Regiomontanus' maximum problem

Johann Müller, alias Regiomontanus, was born in Eastern Prussia near the town of Konigsberg in 1436. He was fond of mathematics and astronomy. He was the first publisher of mathematical and astronomical books for commercial use. For instance, he wrote an accurate table of Sines, a table of position of the sun, moon and planets. His most influential work was his *De triangulis omnimodis*, a work in five sections in which he dealt with the trigonometric and geometric heritage of Ptolemy and the Hindu and Arab scholars. In 1471 Regiomontanus posed the following problem:

> "At which point on the ground does a perpendicularly suspended rod[1] appear largest [that is to say : subtends the greatest visual angle]?"

Let's answer the question using calculus, even though at Regiomontanus' time, it had not yet been created!



In the figure above, let the rod be represented by the vertical line segment AB. Let $OA = a$, $OB = b$ and $OP = x$, where P is the point on the ground at which the angle $\theta = \angle BPA$ is maximum.

Let $\alpha = \angle OPA$ and $\beta = \angle OPB$. We have :

$$\tan\alpha = \tan(\beta + \theta) = \frac{\tan\theta + \tan\beta}{1 - \tan\theta\tan\beta}.$$

Substituting $\tan\beta$ with $\frac{b}{x}$, we get $\tan\alpha = \frac{\tan\theta + \frac{b}{x}}{1 - \tan\theta \times \frac{b}{x}}$. Since $\tan\alpha = \frac{a}{x}$, it yields $\tan\theta = \frac{(a-b)x}{x^2+ab}$.

Let's differentiate this expression in order to find the value of x that maximizes it (since the tangent function is increasing for $0 \leq \theta < \frac{\theta}{2}$). Once it's done, we can easily see that the maximum value is reached at $x = \sqrt{ab}$.

The required point P is thus located at a distance equal to the geometric mean of a and b from O!

Adapted from *Trigonometric Delights* by Eli Maor

Questions

1. Sum up Johann Müller's life in a few words.
2. Justify the formula $\tan\alpha = \frac{a}{x}$.
3. Prove that the derivative of the function that maps x to $\frac{(a-b)x}{x^2+ab}$ is $\frac{(a-b)(ab-x^2)}{(x^2+ab)^2}$.
4. Check that this function has a maximum at $x = \sqrt{ab}$.
5. Explain why θ also has a maximum at $x = \sqrt{ab}$.

[1] rod : tige