Baccalauréat, toutes séries Session de juin 2013

Épreuve de section européenne

The pigeonhole principle

In its simplest form the pigeonhole principle can be stated as follows:

If $n+1$ (or more) objects are to be distributed among n boxes, some box must get at least two of the objects.

Consider the following example :

A lattice point is a point in a coordinate plane for which both coordinates are integers.

To show that one of the line segments connecting five lattice points must pass through some lattice point in the coordinate plane, note that there are four "parity" classes for the coordinates of a lattice point: odd, odd ; odd, even ; even, odd ; and even, even. On the pigeonhole principle two of five lattice points must belong to the same class. This implies that the sum of their x-coordinates and of their y-coordinates are both even numbers, making the midpoint of the segment joining the points a lattice point.

Here is another application of the pigeonhole principle. No matter how a set S of 10 positive integers smaller than 100 is chosen there will always be two completely different selections from S that have the same sum. For example, in the set 3, 9, 14, 21, 26, 35, 42, 59, 63, 76 there are the selections 14, 63, and 14, 21, 42, both of which add up to 77.

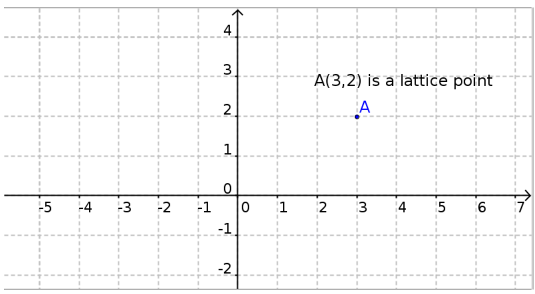
To see why this is always the case, observe that no subset of S can have a sum greater than the 10 largest numbers from 1 to 100 : 90, 91, . . . , 99. These numbers add up to 945, and so the subsets of S can be sorted according to their sum into boxes numbered $1, 2, \ldots, 945$. It happens that the number of subsets of S is 1023. Therefore two of these subsets are in the same box...

Adapted from *The last recreations* by Martin Gardner

Questions

1. First example :

   (a) Give the coordinates of five lattice points with distinct abscissas and ordinates. Can you find a lattice point on a segment joining two of those points ? (You may use the following figure.)



   (b) Explain how the pigeonhole principle argument applies to the situation.

   (c) Explain the argument about the midpoints.

2. Second example : Suppose you choose 6 different numbers in the set 1, 2, 3, . . . , 9, 10.

   (a) What is the greatest value you can get adding these six numbers ?

   (b) There are 63 possibilities to choose a non-empty subset of these six numbers.
   Use the pigeonhole principle to prove that at least two different selections of these six numbers have the same sum.