# Épreuve de section européenne

## Involutions

Investigated by Charles Babbage in the early 19th century, the concept of involution is fundamental in the theory of groups and algebra. Our interest here is to consider involutions in a more humble setting. A function $f$ that maps a set of real numbers onto itself and satisfies on this set the condition :

$$f\left(f\left(x\right)\right)=x$$

or, equivalently,

$$f\left(x\right)=f^{-1}\left(x\right)$$

is called an involution. In other words, an involution is a mapping that coincides with its own inverse. The simplest examples of involutions are :

- reflection (the opposite function) : $f_1\left(x\right)=-x$, $x$ belonging to $\left]-\infty;+\infty\right[$
- inversion (the reciprocal function) : $f_2\left(x\right)=\dfrac{1}{x}$, $x$ belonging to $\mathbf{R}^*=\left]-\infty;+\infty\right[-\left\{0\right\}$
- obviously, $f_3$ such as $f_3\left(x\right)=c-x$ is an involution on $\left]-\infty;+\infty\right[$, where $c$ is an arbitrary real.

Adapted from Joseph Wiener and Will Watkins, *A glimpse into the Wonderland of involutions*

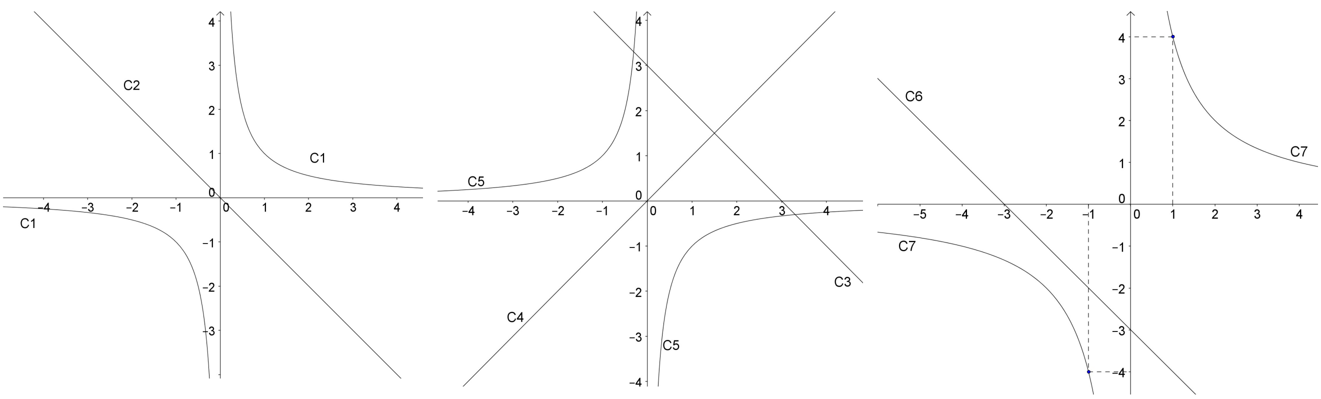
## Questions

1. Check that $f_1$ and $f_2$ are involutions.

2. Explain why $f_3$ is an involution too, whatever the value $c$.

3. Give other examples of involutions.

4. Consider C1, C2, ... C7 the graphs underneath of functions which are involutions.
   a) Do you recognize some of them ?
   b) Those graphs have a common property. Find this property. Comment on it.




5. Have you studied couples of functions which are inverse of each other ?