Épreuve de section européenne

Blankinship's method

The general Bezout's Lemma states that:

For all integers a and b there exist integers s and t such that $\gcd(a,b) = sa + tb$.

In an article in the August-September 1963 issue of the *American Mathematical Monthly*, W.A. Blankinship gave a simple method to produce the integers s and t in Bezout's Lemma and at the same time produce $\gcd(a,b)$:

Given $a > b > 0$ we start with the array $\begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$. Then we continue to add multiples of one row to another row, alternating choice of rows until we reach an array of the form $\begin{bmatrix} 0 & x_1 & x_2 \\ d & y_1 & y_2 \end{bmatrix}$ or $\begin{bmatrix} d & y_1 & y_2 \\ 0 & x_1 & x_2 \end{bmatrix}$.

[The goal is to get a 0 in the first column.]. Then $d = \gcd(a,b) = y_1 a + y_2 b$, therefore $s = y_1$ and $t = y_2$.

Example: First take $a = 35$ and $b = 15$. The starting array is $\begin{bmatrix} 35 & 1 & 0 \\ 15 & 0 & 1 \end{bmatrix}$

Note that $35 = 15 \times 2 + 5$, hence $35 - 15 \times 2 = 5$ (or $35 + 15 \times (-2) = 5$)

So we multiply the second row by -2 and add it to row 1, getting $\begin{bmatrix} 5 & 1 & -2 \\ 15 & 0 & 1 \end{bmatrix}$.

Now $3 \times 5 = 15$ or $15 - 3 \times 5 = 0$, so we multiply the first row by -3 and add it to row 2 (or multiply the first row by 3 and subtract it from row 2). You get $\begin{bmatrix} 5 & 1 & -2 \\ 0 & -3 & 7 \end{bmatrix}$.

Now we can say that: $\gcd(35,15) = 5$ and $5 = 1 \times 35 + (-2) \times 15$. Therefore $s = 1$ and $t = -2$.

Why Blankinship's method works?

Note that just looking at what happens in the first column you see that we are just doing the Euclidean Algorithm, so when one element in column 1 is zero, the other is, in fact, the gcd.

Note that at the start we have $\begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$ and $\begin{matrix} a = 1 \times a + 0 \times b \\ b = 0 \times a + 1 \times b \end{matrix}$. One can show that at every intermediate step $\begin{bmatrix} a_1 & x_1 & x_2 \\ b_1 & y_1 & y_2 \end{bmatrix}$ we always have $\begin{matrix} a_1 = x_1 \times a + x_2 \times b \\ b_1 = y_1 \times a + y_2 \times b \end{matrix}$ and the result follows. I will omit the details.

From *Elementary Number Theory*, by W. Edwin Clark

Questions

1. What do the letters gcd stand for?
2. Use the Euclidean Algorithm to check that $\gcd(55,24) = 1$.
3. Use Blankinship's Method to determine the *s* and *t* in Bezout's Lemma for the values of *a* and *b* given in the previous question.
4. Give examples of maths problems where you use the gcd of two numbers.
5. You want to make two rectangular garden plots next to each other with a common fence, completely around each one, with dimensions in feet being integers. One plot is 180 square feet and the other is 204 square feet. What is the greatest length of the common fence you can make? How much fencing is required at all?