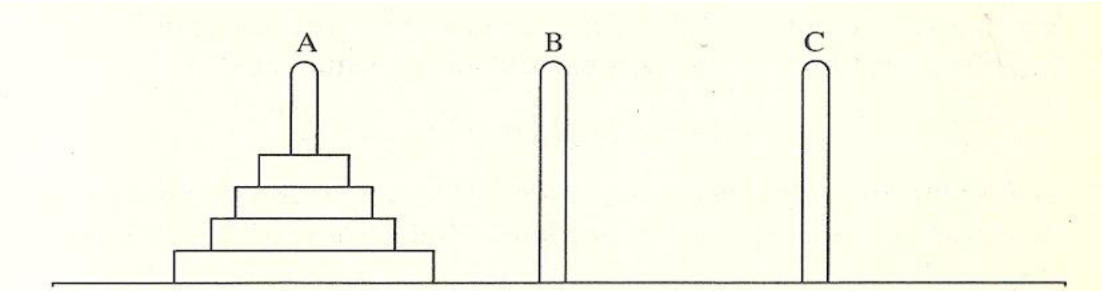
Épreuve de section européenne

The Tower of Hanoï

The classical problem of Hanoi consists of three pegs, A, B and C, together with a tapering tower of concentric rings sitting on the first peg as shown in the figure below. The task is to shift the tower from one peg to another, subject to the following two constraints on the way you can move rings between the pegs:

(a) You may shift only one ring at a time.
(b) No ring may be placed on top of a smaller ring.



Try playing the game with a little tower of three or four rings. You have to find the least number of moves to complete the task.

Next, we need to find the minimum number of moves required for the next n-ring game. The mathematical feature that must be grasped is that, in order to play the n-ring game, you must first, in effect, play the $n-1$-ring game.

For instance, look at your 4-ring game, which is fully representative of the general situation. You cannot move the bottom big ring at all until you have shifted the tower of three rings above it on to another peg - that is, first you play the 3-ring game. Having done this, you can shift the big ring on to the required place in one move, whereupon there is no point in ever moving it again. To complete the procedure, you now must move the tower of three on to the largest ring - that is, you need to play the 3-ring game once more.

If we write a_4 for the minimum number of moves required to shift the tower of four rings, and let a_3 stand for the least number of moves needed for the 3-ring game, the above argument shows us that $a_4 = 1 + 2a_3$. Of course this argument works just as well for the n-ring game.

The story that usually accompanies the Tower of Hanoi (which has 64 rings) is that the monks could move only one ring per day and when they have completed their task some great cataclysm would engulf everything.

Adapted from various sources

Questions

1. Play the 1-ring game and the 2-ring game to find out a_1 and a_2.
2. Explain how "*the above argument shows us that* $a_4 = 1 + 2a_3$ ".
3. "*This argument works just as well for the n-ring game*": deduce from "this argument" the recurrence relation that defines a_n , where a_n stands for the least number of moves required to play the n-ring game to completion.
4. Show that $a_n = 2^n - 1$ for any naturel number n .
5. Do we need to worry about the story accompanying the Tower of Hanoi?