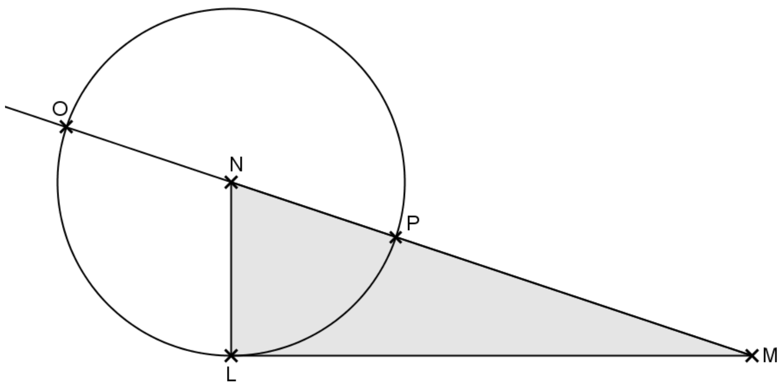
Épreuve de section européenne

Geometry and Quadratic Equations

René Descartes is considered the father of modern philosophy and he was one of the most prominent mathematicians of the Scientific Revolution of the 17th century. One of his largest accomplishments was in the domain of mathematics with the founding of the field of analytic geometry, combination of geometry and algebra. In 1637, in his main work "*la Géométrie*", Descartes first introduced our current practice of representing known quantities with the beginning letters of the alphabet (*a*, *b*, *c*...) and unknown quantities with letters at the end (*x*, *y*, *z*...). He was also the first to use modern exponential notation consistently.

In the first section of "*la Géométrie*", he gave the construction of the positive solution of the quadratic equation $x^2 = ax + b^2$ with a and b positive numbers. He proceeded as follows: first he constructed a right-angled triangle NLM, with one side, LM, equal to the known quantity of b (the square root of b^2), and the other side, LN, equal to half the known quantity a ; then he extended the hypotenuse, MN, to a point O so that NO is equal to NL, the line OM is equal to the required value of the unknown *x*. See the figure below.



Nowadays, it is quite easy to prove that such equations always admit two roots with opposite signs since their product is equal to $-b^2$ which is a negative number. Moreover, the negative root can be deduced from the length PM.

Adapted from *math.uc.denver.edu* website

Questions

1. What is analytic geometry?
2. Applying Descartes's method, draw a sketch and check that the positive solution of the equation $x^2 = 6x + 16$ can indeed be found.
3. Show that the length OM is equal to $\dfrac{a}{2} + \sqrt{\left(\dfrac{a}{2}\right)^2 + b^2}$.
4. We want to prove that Descartes's method is valid.
   a. Prove that the discriminant of the equation $x^2 = ax + b^2$ is equal to $a^2 + 4b^2$ and explain why "such equations always admit two roots".
   b. Work out the positive solution and prove that it is equal to the length OM.