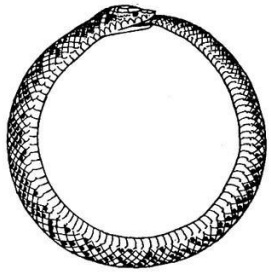
Épreuve de section européenne

Ouroborean Rings

Around 1960 the American mathematician Sherman K. Stein discovered a curious pattern in the Sanskrit nonsense word *yamátárájabhánasalagám*. The composer George Perle told Stein that the stressed (*á*) and unstressed (*a*) syllables form a mnemonic for rythms and correspond to long and short beats. Thus the first three syllables, *ya má tá*, have the rhythm short, long, long. The second to fourth are *má tá rá*, long, long, long – and so on. There are eight possible triplets of long or short rhythms, and each occurs in the nonsense word exactly once.

Stein rewrote the word using 0 for short and 1 for long, getting 0111010001. Then he noticed that the first two digits are the same as the last two, so the string of digits can be bent into a loop, swallowing its own tail Now you can generate all possible sequences of three digits 0 and 1 by moving along the loop one space at a time.

0 1 1 1 0 1 0 0 ...
0 1 1
  1 1 1
    1 1 0
      1 0 1
        0 1 0
          1 0 0
            0 0 0
              0 0 1



I call such sequences *ouroborean rings*, after the mythical serpent Ouroboros, which can eat its own tail.

There is an ouroborean ring for pairs: 0011. It is unique, except for rotations. And for quadruplets, an ouroborean ring is a sequence with eight 0's and eight 1's in a ring so that every possible string of four digits from 0000 to 1111 appears as a series of consecutive symbols (each string of four must then occur exactly once).

From *Cabinet of Mathematical Curiosities*, by Ian Stewart

Questions

1. a. Write down the four possible pairs of 0 and 1.
   b. Check that these pairs appear in the given ouroborean ring 0011.
   c. Using rotations, give another ouroborean ring for pairs.

2. The text says there are eight possible triplets of long or short rhythm. Explain why.

3. Compute the number of different quadruplets of 0 and 1.

We also denote a pair as a 2-tuple, a triplet as a 3-tuple, a quadruplet as a 4-tuple, ... It has been proved that the number r_n of ouroborean rings formed with the *n*-tuples containing the two digits 0 and 1 is: $r_n = 2^{2^{n-1}-n}$ (rings obtained by rotations are considered as one).

4. Using this formula, how many ouroborean rings for pairs are there?

5. a. How many ouroborean rings for triplets of 0 and 1 exist?
   b. The text proposes 01110100 as ouroborean ring for triplets. Find the second one.