Épreuve de section européenne

A strange sum

I recently stumbled upon a video in which two physicists proved that the sum of all the integers is equal to $\frac{-1}{12}$ That's right, no mistyping here: a sum of whole positive numbers is equal to a fraction, and even worse, a negative one ! How could such a thing happen in the rigorous world of maths? Well it has a lot to do with the infinite nature of the sum. Let me give you a quick insight.

First, let's call S the sum we want to calculate:

$$S = 1 + 2 + 3 + 4 + 5 + ...$$

And for the sake of our proof, let's consider two other sums:

$$S_1 = 1 - 1 + 1 - 1 + 1 - 1 + ... \text{ and } S_2 = 1 - 2 + 3 - 4 + 5 - ...$$

Calculating S_1 step by step gives alternatively 1 or 0, depending on whether you stop after an odd or an even number of terms: which one do we chose? We can chose the mean of these two values, which is $\frac{1}{2}$.

Now, let's look at S_2 . We can add that sum to itself, to get $2S_2$. By moving the terms of the second copy of S_2 one number to the right we get :

$$\begin{array}{llllllll} 2S_2 = & 1 & -2 & +3 & -4 & +5 & - ... \\ & & +1 & -2 & +3 & -4 & + ... \end{array}$$

Adding the vertically aligned terms, it yields 1 – 1 + 1 – 1 + 1 – 1 + ... : it is S_1 !

Hence we get $2S_2 = S_1 = \frac{1}{2}$, so $S_2 = \frac{1}{4}$

Finally we calculate $S - S_2 = 4 + 8 + 12 + ... = 4S$, so $S = \frac{-1}{12}$. There's no escaping it!

Adapted from *Bad Astronomy*, a blog by Phil Plait

Questions

1. Why does the author seem so astonished in the first paragraph?
2. Calculate S_1 by grouping terms by pairs in two different ways to show that it can either be equal to 0 or 1. What is your personal opinion about choosing the mean of the two values?
3. The author shows a way to calculate S_2 . What do you get if you apply the same method as in question 2 to do so?
4. Write S and S_2 one below the other and subtract term by term to show that

$$S - S_1 = 4 + 8 + 12 + ...$$

5. Deduce from the equation $S - S_2 = 4S$ that $S = \frac{-1}{12}$.