Episode 04 – Gaussian primes

European section – Season 3

Units and primes

Definition (Units)

Any Gaussian integer is obviously divisible by 1, -1, i and $-i$, as $(-1) \times (-1) = 1$ and $i \times (-i) = 1$. We call these four special numbers the *units* of $\mathbf{Z}[i]$.

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Definition (Gaussian prime)

Any Gaussian integer g is divisible by 1, -1, i, $-i$, g, $-g$, $-ig$ and ig. A *Gaussian* prime is a Gaussian integer that is not divisible by an integer different from these ones.
In other words, a Gaussian integer g is not *prime* if it can be written as a non-trivial product $g = e \times f$, where neither e nor f are units.

Conjugate and norm

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The product of a Gaussian integer $g = a + bi$ and its conjugate $\bar{g} = a - bi$ is a non-negative real number, as

$$g \times \bar{g} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

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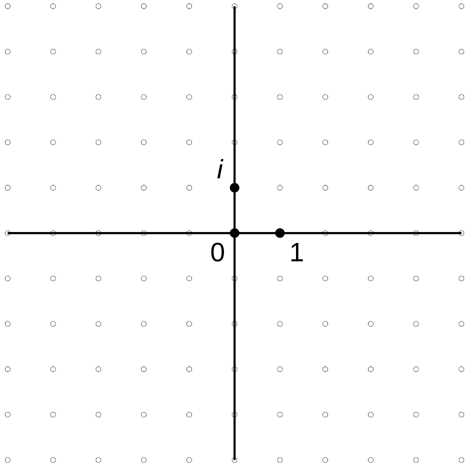
Example

The norm of $g = -3 + 4i$ is $N(g) = (-3)^2 + 4^2 = 25$.

Conjugate and norm



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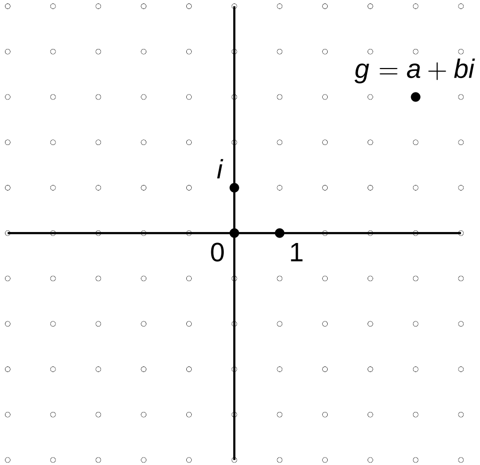

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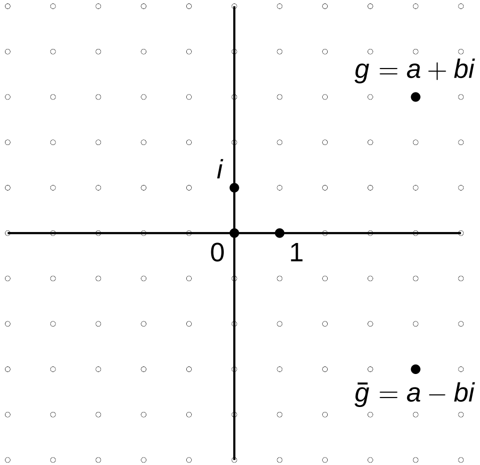

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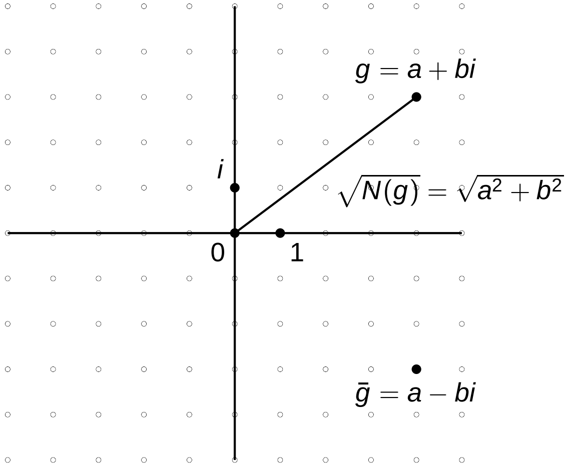

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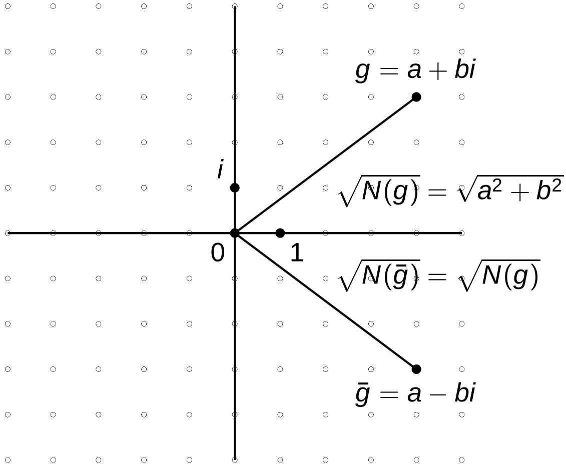

Two important questions

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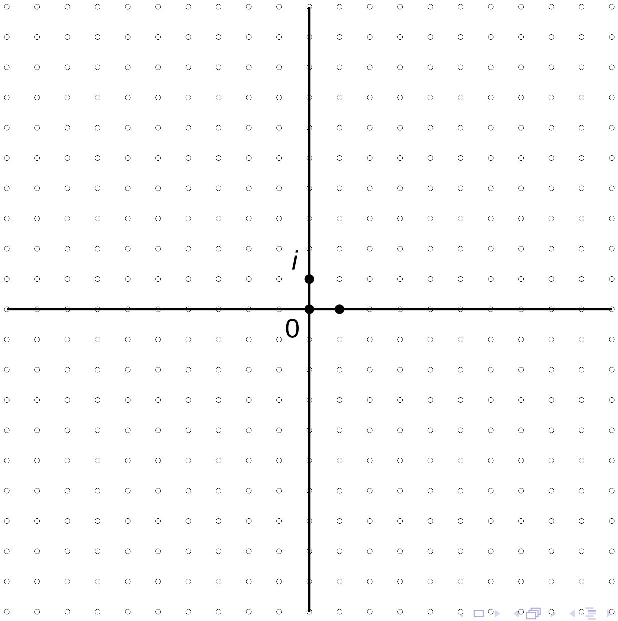
Question 2

Is it still possible to decompose uniquely any Gaussian integer into a product of Gaussian primes ?

Gaussian primes



Primality in $\mathbf{Z}[i]$

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A Gaussian integer $g = a + bi$ is prime if and only if

- *one of a and b is zero and the other is a prime of the form $4n + 3$ or its negative $-(4n + 3)$;*
- *or both are nonzero and the norm of g, $N(g) = a^2 + b^2$ is prime.*

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If $a = 4n + 1$, there exists two integers x and y such that $a = x^2 + y^2 = (x + iy)(x - iy)$. This result is known as "Fermat's theorem on sums of two squares."

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It's obvious that a must be a prime integer, and that if $a = 4n$ or $4n + 2$ it is not prime, as it is divisible by 2.

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It's obvious that a must be a prime integer, and that if $a = 4n$ or $4n + 2$ it is not prime, as it is divisible by 2. If $a = 4n + 1$, Fermat's theorem on sums of two squares assures us that there exists two integers x and y such that $a = x^2 + y^2 = (x + iy)(x - iy)$, so a is composite. Therefore, if a is prime, it must be a prime integer of the form $4n + 3$.

Primality in $\mathbf{Z}[i]$

Continued.

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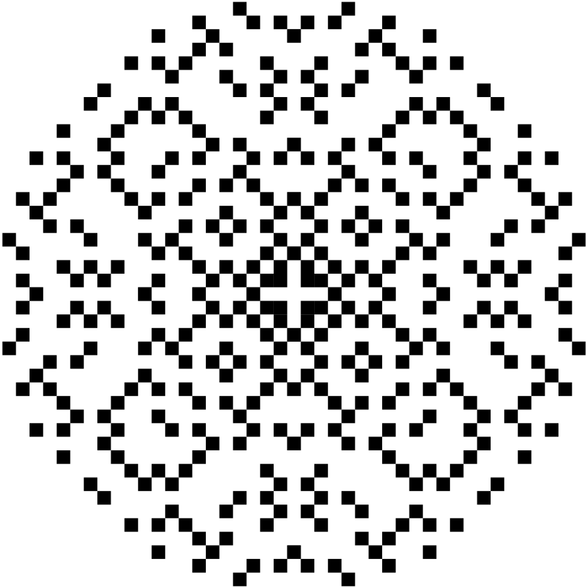
Examples of Gaussian primes

The following Gaussian integers are prime.

$1+i$	$2+i$	3	$3+2i$
$4+i$	$5+2i$	$7+i$	$5+4i$
7	$7+2i$	$6+5i$	$8+3i$
$6+i$	$8+5i$	$9+4i$	

Gaussian primes on a lattice



Unique factorization in $\mathbf{Z}[i]$

Theorem

The set $\mathbf{Z}[i]$ is a unique factorization domain : any Gaussian integer can be written as a product of Gaussian primes, and this decomposition is unique except for reordering or multiplication by units.