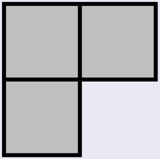
Episode 17 – Tiling with trominoes

European section – Season 3

What is a tromino

Definition

A *tromino* (rhymes with domino) is a shape made up of three 1×1 squares assembled as shown :



Is it possible to tile a deficient n-board with trominoes ?

Study the problem for the following values of n :

1. $n = 2$;
2. $n = 3$;
3. $n = 3k$;
4. $n = 4$;
5. $n = 5$;
6. $n = 7$;
7. $n = 8$;
8. $n = 16$.

A necessary condition

Proposition

A deficient n-board is not tileable with trominoes if n is a multiple of 3. In all other situations, the tiling may be possible.

A necessary condition

Proof.

As the tromino is made of 3 unit squares, a deficient n-board is not tileable with trominoes if $n^2 - 1$ is not divisible by 3.

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As the tromino is made of 3 unit squares, a deficient n-board is not tileable with trominoes if $n^2 - 1$ is not divisible by 3. Consider the division with remainder of n by 3.

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Proof.

As the tromino is made of 3 unit squares, a deficient n-board is not tileable with trominoes if $n^2 - 1$ is not divisible by 3. Consider the division with remainder of n by 3. The remainder is either 0, 1 or 2, so n is equal to $3k$, $3k + 1$ or $3k + 2$.

A necessary condition

Proof.

As the tromino is made of 3 unit squares, a deficient n-board is not tileable with trominoes if $n^2 - 1$ is not divisible by 3. Consider the division with remainder of n by 3. The remainder is either 0, 1 or 2, so n is equal to $3k$, $3k + 1$ or $3k + 2$.

- If $n = 3k$, then $n^2 = 9k^2$ is a multiple of 3. So $n^2 - 1$ is not a multiple of 3 and the deficient $3k$-board is not tileable with trominoes.

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- If $n = 3k + 1$, then $n^2 = 9k^2 + 6k + 1$. So $n^2 - 1 = 9k^2 + 6k = 3(3k^2 + 2k)$ is a multiple of 3 and the deficient $(3k + 1)$-board may be tileable with trominoes.

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As the tromino is made of 3 unit squares, a deficient n-board is not tileable with trominoes if $n^2 - 1$ is not divisible by 3. Consider the division with remainder of n by 3. The remainder is either 0, 1 or 2, so n is equal to $3k$, $3k + 1$ or $3k + 2$.

- If $n = 3k$, then $n^2 = 9k^2$ is a multiple of 3. So $n^2 - 1$ is not a multiple of 3 and the deficient $3k$-board is not tileable with trominoes.
- If $n = 3k + 1$, then $n^2 = 9k^2 + 6k + 1$. So $n^2 - 1 = 9k^2 + 6k = 3(3k^2 + 2k)$ is a multiple of 3 and the deficient $(3k + 1)$-board may be tileable with trominoes.
- If $n = 3k + 2$, then $n^2 = 9k^2 + 12k + 4$. So $n^2 - 1 = 9k^2 + 12k + 3 = 3(3k^2 + 4k + 1)$ is a multiple of 3 and the deficient $(3k + 2)$-board may be tileable with trominoes.

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Proof.

As the tromino is made of 3 unit squares, a deficient n-board is not tileable with trominoes if $n^2 - 1$ is not divisible by 3. Consider the division with remainder of n by 3. The remainder is either 0, 1 or 2, so n is equal to $3k$, $3k + 1$ or $3k + 2$.

- If $n = 3k$, then $n^2 = 9k^2$ is a multiple of 3. So $n^2 - 1$ is not a multiple of 3 and the deficient $3k$-board is not tileable with trominoes.
- If $n = 3k + 1$, then $n^2 = 9k^2 + 6k + 1$. So $n^2 - 1 = 9k^2 + 6k = 3(3k^2 + 2k)$ is a multiple of 3 and the deficient $(3k + 1)$-board may be tileable with trominoes.
- If $n = 3k + 2$, then $n^2 = 9k^2 + 12k + 4$. So $n^2 - 1 = 9k^2 + 12k + 3 = 3(3k^2 + 4k + 1)$ is a multiple of 3 and the deficient $(3k + 2)$-board may be tileable with trominoes.

This condition is necesary but not sufficient : we don't know yet if for $n = 3k + 1$ and $n = 3k + 2$ an actual tiling exists. □

The deficient 5-board

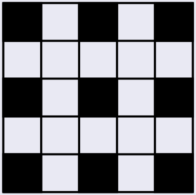
Proposition

If the square (i, j) is removed from the 5-board where either i or j is even, then the resulting shape is not tileable with trominoes. It's easy to check by trial and error that all other deficient 5-boards are tileable.

The deficient 5-board

Proof.

Form a kind of checkerboard design by marking each of the nine squares $(1,1)$, $(1,3)$, $(1,5)$, $(3,1)$, $(3,3)$, $(3,5)$, $(5,1)$, $(5,3)$, $(5,5)$.



and assume that one of the 16 unmarked squares has been removed from the board.

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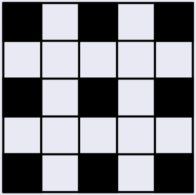

and assume that one of the 16 unmarked squares has been removed from the board.

Then a proposed tiling of the deficient board must contain one tromino for each of the 9 marked squares,

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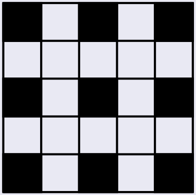

and assume that one of the 16 unmarked squares has been removed from the board.
Then a proposed tiling of the deficient board must contain one tromino for each of the 9 marked squares, so that tiling must have area at least $9 \times 3 = 27$,

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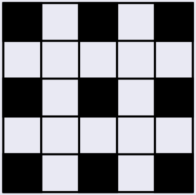
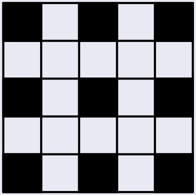


and assume that one of the 16 unmarked squares has been removed from the board.
Then a proposed tiling of the deficient board must contain one tromino for each of the 9 marked squares, so that tiling must have area at least $9 \times 3 = 27$, which is absurd since the deficient board is 24.

The deficient 5-board

Proof.

Form a kind of checkerboard design by marking each of the nine squares $(1,1)$, $(1,3)$, $(1,5)$, $(3,1)$, $(3,3)$, $(3,5)$, $(5,1)$, $(5,3)$, $(5,5)$.



and assume that one of the 16 unmarked squares has been removed from the board.

Then a proposed tiling of the deficient board must contain one tromino for each of the 9 marked squares, so that tiling must have area at least $9 \times 3 = 27$, which is absurd since the deficient board is 24. Thus all 16 of the unmarked squares are bad. □

The deficient 2^k-board

Theorem

Any deficient 2^k-board with $k \in N^\star$ is tileable with trominoes.

The deficient 2^k-board

Proof.

This proof is a famous example of proof by induction. We will first prove that it's true for $k = 1$ and then that if it's true for one value k, it's also true for $k + 1$.

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First, it's obvious that the deficient 2-board is tileable with trominoes,

The deficient 2^k-board

Proof.

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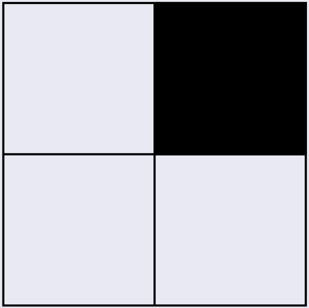
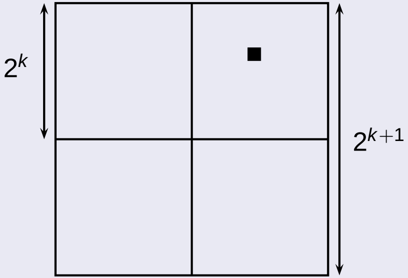
First, it's obvious that the deficient 2-board is tileable with trominoes, as it is a tromino.



The deficient 2^k-board

Proof. (Continued)

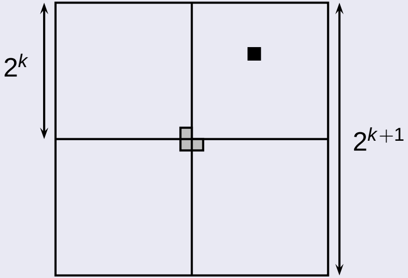
Now, suppose that the deficient 2^k-board is tileable with trominoes, and consider the deficient 2^{k+1}-board. This board can be cut into four quadrants, each one a 2^k-board. The missing square has to be in one of the four quadrant.



The deficient 2^k-board

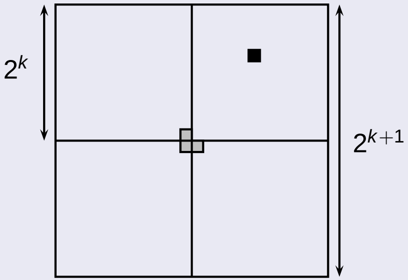
Proof. (continued)

So from our induction hypothesis, that quadrant is tileable with trominoes. We are left with three quadrants to tile. We can then place one tromino at the center of the 2^{k+1}-board so that each of its three squares is in a different quadrant.



The deficient 2^k-board

Proof. (continued).



We must now tile three deficient 2^k-boards, which we know is possible from our induction hypothesis. So the deficient 2^{k+1}-board is tileable with trominoes. As the property is true for $k = 1$, it's also true for $k = 2$, $k = 3$, and so on for all natural values of k. □

Other values of n

Theorem

The deficient n-board is always tileable with trominoes, except for $n = 3k$, in which case it is not, and for $n = 5$, in which case the answer depends on the position of the missing square.

This result is a consequence of a more general one proved by J. Marshal Ash and Solomon W . Golomb in the article *Tiling Deficient Rectangles with Trominoes*, Mathematics Magazine, October 17, 2003.