Episode 18 – The Tower of Hanoi

European section – Season 3

The legend

The puzzle was invented by the French mathematician *Édouard Lucas* in 1883. It's based on a Vietnamese or Indian legend about temple which contained a large room with three time-worn posts in it surrounded by 64 golden disks.

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According to the legend, when the last move of the puzzle is completed, the world will end.

Problem 1

Find a solution for 2 disks, for 3 disks and for 4 disks. Count the number of moves needed and try to lower it as much as possible.

Problem 2

Where must we move the first disk ?
Is their a simple algorithm to solve the puzzle ?

Problem 3

What is the lowest number of moves for a n disks tower ?

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- If there is no tower in the chosen direction, move it to the opposite end.
- When the turn is to move the non-smallest piece, there is only one legal move.

The lowest number of moves

Theorem

The lowest number of moves necessary to move a n disks tower is $2^n - 1$.

The lowest number of moves : a proof

Proof.

Let's prove this result by induction.

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Basis : First, consider the 1-disk situation. Then, the lowest number of moves is obviously $1 = 2^1 - 1$, so the property is true.

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$$(2^k-1)+1+(2^k-1)=2\times 2^k-1=2^{k+1}-1.$$

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Conclusion : Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that the property holds for all natural n.

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